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THE ENERGY ASPECT OF THE RECIPROCAL INTERACTIONS OF PAIRS OF TWO DIFFERENT VIBRATION MODES OF A CLAMPED ANNULAR PLATE

W. P. RDZANEK JR

Acoustics Department, University of Rzeszów, ul. Rejtana 16*A*, 35-310 *Rzeszów, Poland.*
E-mail: *wprdzank@univ.rzeszow.pl*

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The standardized mutual active and reactive sound power of a clamped plate, representing the energy aspect of the reciprocal interactions of two different *in vacuo* modes, has been computed. It was assumed that the vibrations are axisymmetric, elastic and time harmonic, the plate's transverse deflection is small as compared with the plate's size, and that the vibration velocity is small as compared with the acoustic wavenumber generated. The Kirchhoff–Love theory of a perfectly elastic plate was used. The integral formulae for the mutual sound power were transformed into their Hankel representations which made possible their subsequent computation. A closed path integral was used to express the integral in its Hankel representation to compute the mutual active sound power. The asymptotic stationary phase method was used to compute the two magnitudes, i.e., the mutual active and reactive sound power. The results obtained are the asymptotic formulae valid for the acoustically fast waves. The oscillating as well as the non-oscillating terms have been identified in the formulae to make possible their further separate analysis. The availability of the asymptotic formulae makes possible some fast numerical computations of the mutual sound power. Moreover, the formulae presented herein, together with those for the individual modes known from the literature, make a complete basis for further computations of the total sound power of the plate's damped and forced vibrations in fluid.

1. INTRODUCTION

It is necessary to determine the total sound power, active and reactive, to correctly design any acoustic system. For that purpose both methods, experimental and purely theoretical, are usually used. However, a number of problems have not yet been considered analytically, because of their complexity even for considerably simple surface sound sources as, e.g., the square piston [1] with numerous simplifying assumptions made. In the case of annular plates [2–5], the analytical considerations of the problem of sound radiation are even more complex.

Many theoretical investigations are still performed to find solutions to the problem of particular vibrating surface sound sources, because existing solutions are highly general and only allow rough approximations of the magnitudes describing sound radiation of the sources. In order to analyze the energy aspect of the sound source, it is absolutely necessary to find the active and reactive sound power for individual *in vacuo* modes as well as for the pairs of two different *in vacuo* modes. The complete theoretical analysis of sound radiation is possible only if all the magnitudes are known. This makes possible the computation of the total sound power of any forced and damped surface sources in a fluid.

The problem of free vibrations of various kinds of surface sound sources has been adequately examined for circular and annular plates (cf. references [6–12]). The sound pressure distribution of an annular plate was determined by computing the sound energy stream and presented in references [13–15]. Some analytical formulations for the plate in the form of multiple expansion series have been reached in reference [2]. The active and reactive sound power of a clamped annular plate has been given in reference [5]. An asymptotic–impedance approach was employed to find the effective damping parameter of a circular plate in reference [16]. The mutual sound power of plate reference [17], and the total sound power of a forced and damped plate in a fluid [18] have also been presented in the literature.

The analytical formulations for the active and reactive mutual sound power of some pairs of two different *in vacuo* modes of an annular plate have not been presented yet. In this paper, an asymptotic–impedance approach has been employed to determine those formulations. The problem of free vibrations has been solved using the Kirchhoff–Love theory of a perfectly elastic homogeneous plate. The path integral computing method in the complex variable plane is used to express the integral for the active mutual sound power in the form computable by means of the stationary phase method. Further, the stationary phase method is employed immediately to determine the formula for the reactive mutual sound power. This leads to some asymptotic formulae valid for the acoustically fast waves, i.e., when the acoustic wavenumber is greater than the structural wavenumber. The results obtained are illustrated graphically in the form of several plotted curves of the mutual sound power for some sample pairs of two different *in vacuo* modes of the plate in the dimensionless frequency domain. Moreover, the asymptotic formulae presented herein together with those for the sound power of the individual *in vacuo* modes (given in reference [5]) can make the complete basis for a full, purely theoretical, analysis of the energy aspect of a forced and damped annular plate in a fluid.

2. SOUND RADIATION

The sound power of the reciprocal interactions of pairs of two different *in vacuo* modes of a source is necessary for determining its total sound power. The Kirchhoff–Love theory of a perfectly elastic plate is used to analyze some time-harmonic and axisymmetric processes. The plate is clamped in a coplanar and perfectly rigid baffle, i.e., the vibration velocity distribution is equal to zero on all the baffle's surface. The plate's radii are r_1 (internal) and r_2 (external) respectively.

2.1. THE VELOCITY DISTRIBUTION OF A PLATE

A detailed analysis of the free vibrations of a clamped annular plate has been presented in the literature several times (e.g., references [5, 11, 12]). All the derivations of the plate's motion presented below are quoted from reference [5]. The plate's equation of motion of some time-harmonic and axisymmetric processes can be presented in the amplitude form

$$(k_n^{-4}\nabla_r^4 - 1)\eta_n(r) = 0, \tag{1}$$

where r is the plate's radius, $n = 0, 1, 2, \ldots$ is the plate's individual mode index, k_n ($k_n^2 = \omega_n\sqrt{\varrho h/D}$) is the structural wavenumber, ω_n is the nth eigenfrequency, ϱ, h, E are the density, thickness, and the Young modulus of the plate, respectively, and $D = Eh^3/[12(1-\nu^2)]$ is the plate's stiffness, ν is the Poisson ratio of its material, $\nabla_r^4 = (\nabla_r^2)^2$

and $\nabla_r^2 = \partial^2/\partial r^2 + r^{-1}\partial/\partial r$, the plate's transverse deflection amplitude of its nth individual mode is predicted as

$$\eta_n(r)/A_n = \mathrm{J}_0(k_n r) + B_n \mathrm{I}_0(k_n r) - C_n \mathrm{N}_0(k_n r) - D_n \mathrm{K}_0(k_n r). \tag{2}$$

A_n, B_n, C_n, D_n are the constants, and J_0, I_0, N_0, K_0 are the functions of zero order: Bessel, modified Bessel, Neumann and McDonald respectively. The clamped plate implies the following boundary conditions:

$$\eta_n(r_1, t) = 0, \quad \eta_n(r_2, t) = 0, \quad \partial\eta_n(r, t)/\partial r|_{r=r_1} = 0, \; \partial\eta_n(r, t)/\partial r|_{r=r_2} = 0, \tag{3}$$

where t is a time variable. Solution (2) inserted into the boundary conditions (3) provides the equation system

$$\begin{bmatrix} \mathrm{J}_0(x_n) & \mathrm{I}_0(x_n) & -\mathrm{N}_0(x_n) & -\mathrm{K}_0(x_n) \\ \mathrm{J}_0(sx_n) & \mathrm{I}_0(sx_n) & -\mathrm{N}_0(sx_n) & -\mathrm{K}_0(sx_n) \\ -\mathrm{J}_1(x_n) & -\mathrm{I}_1(x_n) & \mathrm{N}_1(x_n) & \mathrm{K}_1(x_n) \\ -\mathrm{J}_1(sx_n) & -\mathrm{I}_1(sx_n) & \mathrm{N}_1(sx_n) & \mathrm{K}_1(sx_n) \end{bmatrix} \cdot \begin{bmatrix} 1 \\ B_n \\ C_n \\ D_n \end{bmatrix} = \mathbf{0}, \tag{4}$$

where $s = r_2/r_1 > 1$ is the plate's geometric parameter, $x_n = k_n r_1$ are the roots, or eigenvalues, of the plate's frequency equation

$$[sN(sx_n) - N(x_n)]\,[sT(sx_n) - T(x_n)] = [sS(sx_n) - S(x_n)]\,[sR(sx_n) - R(x_n)], \tag{5}$$

where the following denotations are used:

$$S(x) = \mathrm{J}_1(x)\mathrm{I}_0(x) + \mathrm{J}_0(x)\mathrm{I}_1(x), \qquad T(x) = \mathrm{N}_1(x)\mathrm{I}_0(x) + \mathrm{N}_0(x)\mathrm{I}_1(x),$$

$$N(x) = \mathrm{J}_1(x)\mathrm{K}_0(x) - \mathrm{J}_0(x)\mathrm{K}_1(x), \qquad R(x) = \mathrm{N}_1(x)\mathrm{K}_0(x) - \mathrm{N}_0(x)\mathrm{K}_1(x). \tag{6}$$

The derivation of equation (5) has been shown in reference [5] and was reached by setting the main determinant of the equation system (4) to zero.

Equation system (4) provides three of the four constants

$$B_n = sx_n\,(R(sx_n)N(x_n) - R(x_n)N(sx_n))/(sR(sx_n) - R(x_n)), \tag{7}$$

$$\mathrm{C}_n = (s\,S(sx_n) - S(x_n))/sT(sx_n) - T(x_n), \tag{8}$$

$$D_n = sx_n\,(T(sx_n)S(x_n) - T(x_n)S(sx_n))/(sT(sx_n) - T(x_n)). \tag{9}$$

The fourth constant

$$A_n = g_n^{-1}\sqrt{(s^2 - 1)/2} \tag{10}$$

is derived from the standardization condition

$$\int_{r_1}^{r_2} \eta_n^2(r) r\,\mathrm{d}r = r_1^2(s^2 - 1)/2,$$

where

$$g_n = \sqrt{s^2 G_0^2(sx_n) - G_0^2(x_n)}, \; G_\mu(x) = \mathrm{J}_\mu(x) - C_n \mathrm{N}_\mu(x), \; x \in \{x_n, sx_n\}, \; \mu \in \{0, 1\}.$$

2.2. THE SOUND PRESSURE DISTRIBUTION

The vibrating surface sound source radiates the sound pressure into a hemisphere $z \geqslant 0$, filled with a loss-less gaseous medium of rest density ϱ_0 and wave propagation velocity c. The distance between the two different points, $M(\mathbf{r})$ (in the soundfield) and $M_0(\mathbf{r}_0)$ (at the source), is $|\mathbf{r} - \mathbf{r}_0| = [R^2 + r_0^2 - 2Rr_0 \cos(\mathbf{r}, \mathbf{r}_0)]^{1/2}$, where $\mathbf{r}$, $\mathbf{r}_0$ are the radius vectors of the points and $\cos(\mathbf{r}, \mathbf{r}_0) = \sin\vartheta \cos(\varphi - \varphi_0)$ in spherical co-ordinates. In all the theoretical analyses presented herein, the locations of the field points are presented in the spherical co-ordinates R, ϑ, φ and the locations of the source points are presented in the polar co-ordinates r_0, φ_0 in plane $z_0 = 0$.

The basis of the analysis is the integral formula by Rayleigh for sound pressure distribution

$$p(\mathbf{r}) = \frac{-\mathrm{i}k\varrho_0 c}{2\pi} \int_{S_0} v(\mathbf{r}_0) \frac{\mathrm{e}^{\mathrm{i}k|\mathbf{r} - \mathbf{r}_0|}}{|\mathbf{r} - \mathbf{r}_0|} \, \mathrm{d}S_0, \tag{11}$$

where $v(\mathbf{r}_0)$ is the plate's normal velocity distribution, S_0 is the plate's area, k is the acoustic wavenumber, ϱ_0, c are the density of air and sound velocity in air respectively. Equation (11) cannot be integrated in its current form because its integrand comprises an exponential component and therefore it is necessary to express the sound pressure in its Hankel representation (cf. Reference [19])

$$p(r, z) = \frac{k\varrho_0 c}{2\pi} \int_0^{+\infty} \int_0^{2\pi} \tilde{W}(\tau) \, \mathrm{e}^{\mathrm{i}[\tau r \cos(\varphi - \alpha) + \gamma z]} \, \mathrm{d}\alpha \frac{\tau \, \mathrm{d}\tau}{\sqrt{k^2 - \tau^2}}, \tag{12}$$

where τ is a complex acoustic wavelength (cf. Figure 1), r, φ, z are the polar co-ordinates, α is an angle variable,

$$\tilde{W}(\tau) = \int_{r_2}^{r_1} v(r_0) \mathrm{J}_0(\tau r_0) r_0 \, \mathrm{d}r_0 \tag{13}$$

is a characteristic function of the vibrating plate, and the complex parameter $\gamma = \gamma' + \mathrm{i}\gamma'' = \sqrt{k^2 - \tau^2}$ represents the propagation of both waves—homogeneous and non-homogeneous—in the direction of increasing positive values of z. The sound pressure of any point of half-space $z \geqslant 0$, produced by the axisymmetric vibrations of the source, is

$$p(r, z) = k\varrho_0 c \int_0^{+\infty} \tilde{W}(\tau) \mathrm{J}_0(\tau r) \frac{\mathrm{e}^{\mathrm{i}\gamma z}}{\gamma} \tau \, \mathrm{d}\tau. \tag{14}$$

The sound pressure radiated depends on time as follows $p(r, z, t) = p(r, z)\mathrm{e}^{-\omega t}$. Completing the term $\mathrm{e}^{\mathrm{i}\gamma z}$ with this dependence gives $\mathrm{e}^{-\mathrm{i}\omega t}\mathrm{e}^{\mathrm{i}\gamma z} = \mathrm{e}^{\mathrm{i}(\gamma' z - \omega t)}\mathrm{e}^{-\gamma'' z}$. Assuming that $\gamma', \gamma'' > 0$, it is obvious that the term $\mathrm{e}^{\mathrm{i}(\gamma' z - \omega t)}$ is valid for some planar homogeneous waves propagated in the direction of increasing values of z. On the other hand, the term $\mathrm{e}^{-\gamma'' z}$ characterizes the non-homogeneous wave, whose damping increases proportionally with the value of z. The integrand in equation (14) comprises the product of functions characterizing both kinds of waves—cylindric of zero order $\mathrm{J}_0(\tau r)$ and planar $\mathrm{e}^{\mathrm{i}\gamma z}$. The integration is performed within the limits $(0, \infty)$, along the real axis in the plane of complex variable $\tau = \tau' + \mathrm{i}\tau''$ (cf. Figure 1). The branch point of term $\gamma = \sqrt{k^2 - \tau^2}$ in equation (14) is $\tau = k$, and its phase is equal to zero within the limits $(0, k)$ or $\pi/2$ within the limits (k, ∞) and $\sqrt{k^2 - \tau^2} = \mathrm{i}\sqrt{\tau^2 - k^2}$. The integration is performed along the bottom side of the branch point.

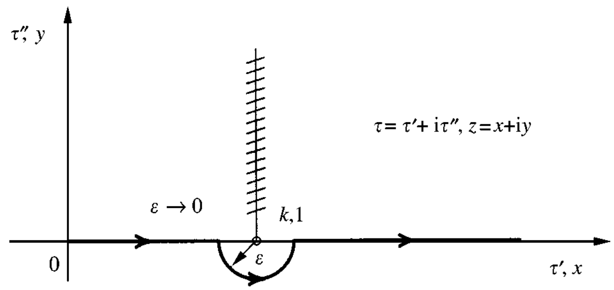

Figure 1. The integration path in a plane of two complex variables τ and z used in equations (14), (20) and (21), (24), (28), (30), (46) (48b), (51) respectively (cf., reference [23]).

3. THE SOUND POWER OF THE RECIPROCAL INTERACTIONS

The reciprocal interactions of pairs of two different *in vacuo* modes of a plate are transmitted via the plate's material. The two values of the standardized mutual sound power, active and reactive, are positive for some frequencies and negative for the other ones. The positive values result from the fact that the plate absorbs some sound energy via its nth mode transmitted via its mth mode, whereas, the negative values result from the fact that some sound energy is lost by the plate via its nth mode and absorbed via its mth mode. For the frequencies tending towards infinity or zero ($\omega \to \infty$, $\omega \to 0$) both magnitudes vanish. For some other frequencies, the amplitude of the mutual sound power is also equal to zero. This shows that for those frequencies there are no reciprocal interactions of some pairs of two different *in vacuo* modes. The additional sound power, lost by the vibrating plate via its nth mode to overcome the resistance produced by its mth mode is understood as the mutual sound power Π_{nm} of a pair of two different *in vacuo* modes, the nth and the mth, further referred to as nm. The additional power can be expressed by the well-known integral formula

$$\Pi_{nm} = \tfrac{1}{2} \int_S p_m v_n^* \, \mathrm{d}S, \tag{15}$$

where S is a closed surface, enclosing the radiation source, $v_n^* = \mathrm{i}\omega_n \eta_n$ is the nth conjugate mode shape of the plate, ω_n is the plate's nth eigenfrequency, $\mathrm{i} = \sqrt{-1}$, p_m is the sound pressure emitted by the plate via its mth *in vacuo* mode. The integration in equation (15) is performed over surface S. The integral computed over the hemisphere enclosing the plate vanishes with an increase in its radius based on Sommerfield's radiation condition [19]. The integral computed over the planar and perfectly rigid surface of the baffle is equal to zero, because the value of its velocity distribution is also equal to zero. Only the integral computed over the plate's surface is non-zero. Therefore, the integration in equation (15) is performed over the whole surface of the plate, i.e., within the limits $[r_1, r_2]$. The integration carried out in the polar co-ordinates within the limits $[0, 2\pi]$ of angle variable φ is trivial, provided that the processes considered herein are axisymmetric.

It is worth noting that such magnitudes as the sound power and the sound impedance are equivalent. The sound impedance was considered in the case of the vibrating circular plate [17, 20], and can be expressed by virtue of its definition (cf., reference [21])

$$Z_{nm} = X_{nm} - \mathrm{i}Y_{nm} = [4\langle |v_n|^2\rangle \langle |v_m|^2\rangle]^{-1/2} \int_S p_m v_n^* \, \mathrm{d}S, \tag{16}$$

where X_{nm} is the mutual sound resistance, Y_{nm} is the mutual sound reactance, $\langle |v_n|^2 \rangle = (2S)^{-1} \int_S v_n^2(r)\,\mathrm{d}S$ is the value of time-averaged velocity square of the plate's nth *in vacuo* mode. In this paper an analysis of the sound power Π_{nm} has been presented.

3.1. THE INTEGRAL FORMULAE

The sound pressure has been presented in the Hankel representation in equation (14) to make possible its further computation. The equation can also be used for deriving the sound pressure $p_m(r)$ radiated by the mth individual mode of the plate, when $z = 0$. It is convenient to define the function characterizing the *mth* mode shape as

$$\tilde{W}_m(\tau) = \int_{r_1}^{r_2} v_m(r_0)\,\mathrm{J}_0(\tau r_0) r_0\,\mathrm{d}r_0 \tag{17}$$

where r_0 is the distance between the plate's point and the plate's center. Computing the integral in equation (17) provides its elementary form

$$\tilde{W}_m(\tau) = -\,\mathrm{i}\omega_m A_m\,(r_1^2/\beta)\,2\delta_m^2 \tilde{\psi}_m(\tau)/(\delta_m^4 - (\tau/k)^4), \tag{18}$$

where $\delta_m = x_m/\beta$, $\beta = kr_1$ is a dimensionless frequency parameter and

$$\tilde{\psi}_m(\tau) = \delta_m[sG_1(sx_m)\mathrm{J}_0(s\tau r_1) - G_1(x_m)\mathrm{J}_0(\tau r_1)] \\ -(\tau/k)[sG_0(sx_m)\mathrm{J}_1(s\tau r_1) - G_0(x_m)\mathrm{J}_1(\tau r_1)]. \tag{19}$$

Equations (14) and (18) make possible expressing the sound pressure radiated by the mth individual mode, for $z = 0$, in the form of

$$p_m(r) = k\varrho_0 c \int_0^{+\infty} \tilde{W}_m(\tau)\mathrm{J}_0(\tau r)\gamma^{-1}\tau\,\mathrm{d}\tau \tag{20}$$

or, with replacements $\tau = kx$ and $\gamma = \sqrt{k^2 - \tau^2} = k\sqrt{1 - x^2}$, as

$$p_m(r) = k^2 \varrho_0 c \int_0^{+\infty} W_m(x)\mathrm{J}_0(krx)\,\frac{x\,\mathrm{d}x}{\sqrt{1 - x^2}}, \tag{21}$$

where

$$W_m(x) = -\,\mathrm{i}\omega_m A_m\,(r_1^2/\beta)\,2\delta_m^2 \psi_m(x)/(\delta_m^4 - x^4), \tag{22}$$

$$\psi_m(x) = \delta_m[sG_1(sx_m)\mathrm{J}_0(s\beta x) - G_1(x_m)\mathrm{J}_0(\beta x)] \\ -x[sG_0(sx_m)\mathrm{J}_1(s\beta x) - G_0(x_m)\mathrm{J}_1(\beta x)]. \tag{23}$$

Inserting equation (21) into equation (15) provides the mutual sound power in its Hankel representation

$$\Pi_{nm} = \pi\varrho_0 c k^2 \int_0^{+\infty} W_m(x) W_n^*(x)\,\frac{x\,\mathrm{d}x}{\sqrt{1 - x^2}}. \tag{24}$$

The reference sound power

$$\Pi_{nm}^{(\infty)} = \sqrt{\Pi_n^{(\infty)}\Pi_m^{(\infty)}} \tag{25}$$

consists of the active components only and

$$\Pi_n^{(\infty)} = \lim_{k\to\infty} \Pi_n = \frac{\varrho_0 c}{2}\int_S v_n v_n^* \,\mathrm{d}S = \pi\varrho_0 c\omega_n^2 A_n^2 r_1^2 g_n^2 . \tag{26}$$

Inserting equation (26) into equation (25) gives

$$\Pi_{nm}^{(\infty)} = \pi\varrho_0 c\omega_n\omega_m A_n A_m r_1^2 g_n g_m, \tag{27}$$

which is used to standardize the mutual sound power from equation (24)

$$\mathscr{P}_{nm} = \mathscr{P}_{a,nm} - \mathrm{i}\mathscr{P}_{r,nm} = \Pi_{nm}/\Pi_{nm}^{(\infty)} = \int_0^{+\infty} \Psi_n(x)\Psi_m(x)\frac{x\,\mathrm{d}x}{\sqrt{1-x^2}}, \tag{28}$$

where $\mathscr{P}_{a,nm}$ and $\mathscr{P}_{r,nm}$ are the standardized mutual sound power, active and reactive, respectively, and

$$\Psi_m(x) = (2\delta_m^2/g_m)\,\psi_m(x)/(x^4 - \delta_m^4). \tag{29}$$

The integration in equation (28) is performed in the plane of complex variable x along the path shown in Figure 1. The phase of term $\sqrt{1-x^2}$ in equation (28) is equal to zero within the limits (0, 1) or $\pi/2$ within the limits $(1, \infty)$ and $\sqrt{1-x^2} = \mathrm{i}\sqrt{x^2-1}$, which is identical to the phase of the analogous term in equation (14).

3.2. THE ASYMPTOTIC FORMULATIONS

Limiting the integration in equation (28), along the path shown in Figure 1, to the finite limits $0 \leqslant x \leqslant 1$ provides the standardized active mutual sound power

$$\mathscr{P}_{a,nm} = \int_0^1 \Psi_n(x)\Psi_m(x)\frac{x\,\mathrm{d}x}{\sqrt{1-x^2}}. \tag{30}$$

The integral in equation (30) can be expressed using another integral computed within the same limits but along a different path shown in Figure 2. The method was used earlier by Levine and Leppington [16] or by Rdzanek [17] dealing with circular plates. It is necessary to introduce function F in the plane of complex variable z as follows:

$$\begin{aligned} F(z) = s\{ &-\delta_n\delta_m[G_1(sx_n)G_1(x_m) + G_1(x_n)G_1(sx_m)]\mathrm{J}_0(\beta z) \\ &+ [\delta_n G_1(sx_n)G_0(x_m) + \delta_m G_0(x_n)G_1(sx_m)]z\mathrm{J}_1(\beta z) \\ &+ s\delta_n\delta_m G_1(sx_n)G_1(sx_m)\mathrm{J}_0(s\beta z)\}\,\mathrm{H}_0^{(1)}(s\beta z) \\ &+ \delta_n\delta_m G_1(x_n)G_1(x_m)\mathrm{J}_0(\beta z)\mathrm{H}_0^{(1)}(\beta z) \\ &+ s\{-s[\delta_n G_1(sx_n)G_0(sx_m) + \delta_m G_0(sx_n)G_1(sx_m)]\mathrm{J}_0(s\beta z) \end{aligned}$$

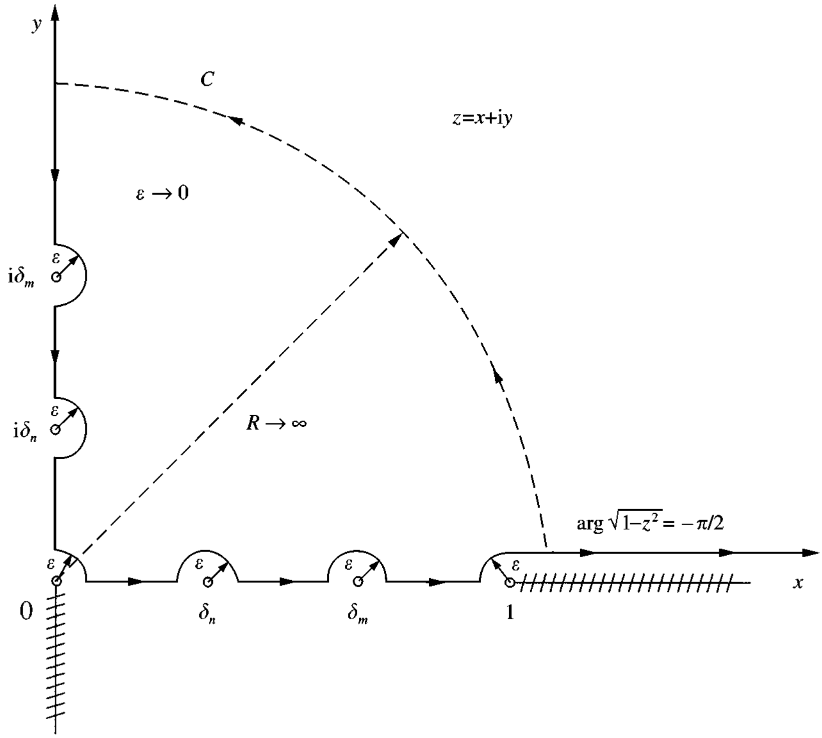

Figure 2. The integration path C in equations (32), (33), (35), (41) (cf., references [4, 5, 16]). This path is used to derive the asymptotic formula for the mutual active sound power only.

$$+ [\delta_n G_1(x_n) G_0(sx_m) + \delta_m G_0(sx_n) G_1(x_m)] \mathrm{J}_0(\beta z)$$
$$- z[G_0(x_n) G_0(sx_m) + G_0(sx_n) G_0(x_m)] \mathrm{J}_1(\beta z)$$
$$+ zs G_0(sx_n) G_0(sx_m) \mathrm{J}_1(s\beta z)\} z\mathrm{H}_1^{(1)}(s\beta z)$$
$$+ \{G_0(x_n) G_0(x_m) z J_1(\beta z) - [\delta_n G_1(x_n) G_0(x_m)$$
$$+ \delta_m \mathrm{G}_0(x_n) G_1(x_m)] \mathrm{J}_0(\beta z)\} z\mathrm{H}_1^{(1)}(\beta z), \tag{31}$$

where $\mathrm{Re}\, F(x) = \psi_n(x)\psi_m(x)$. Function F in equation (31) must be analytical and regular within and along the path C and conditions $k > k_n$, $k > k_m$, $n \neq m$ must be satisfied. Then, based on the Cauchy theorem, the following equality must be satisfied (cf., reference [17]):

$$\oint_C \frac{zF(z)\,\mathrm{d}z}{\sqrt{1-z^2}(z^4 - \delta_n^4)(z^4 - \delta_m^4)} = 0, \tag{32}$$

where the integral is computed along path C, which is closed above the branch cut within the limits $(1, +\infty)$ (cf., Figure 2) as a consequence of using the first kind Hankel function. The Cauchy theorem on residue results in the symbolic form of equation (32)

$$\int_0^1 + \int_1^\infty + \int_{C_R} + \int_{+\mathrm{i}\infty}^0 = \pi \mathrm{i} \sum_{j=1}^4 \mathrm{Re}_{z_j} s, \tag{33}$$

where $z_j \in \{\delta_n, \mathrm{i}\delta_n, \delta_m, \mathrm{i}\delta_m\}$ are the singular points (first order poles), whereas integrals $⨍_0^1$, $⨍_{\mathrm{i}\infty}^0$ are interpreted as the Cauchy principal values. The integration around the branch points $z = 0$, 1, and along the big circle, when $R \to \infty$, gives no contribution to (33)—i.e., $\lim_{R\to\infty}\int_{C_R} = 0$. Point $z = 1$ is the branch point of term $\sqrt{1-z^2}$, and $z = 0$ is the branch point of the zero order Hankel function, which has a logarithmic singularity for $|z| \to 0$. The following auxiliary functions are employed to compute the residue in the first order poles:

$$\mathscr{F}_1(z) = \frac{zF(z)}{\sqrt{1-z^2}(z+\delta_n)(z^2+\delta_n^2)(z^4-\delta_m^4)} \quad \text{for } z = \delta_n, \tag{34a}$$

$$\mathscr{F}_2(z) = \frac{zF(z)}{\sqrt{1-z^2}(z+\mathrm{i}\delta_n)(z^2-\delta_n^2)(z^4-\delta_m^4)} \quad \text{for } z = \mathrm{i}\delta_n, \tag{34b}$$

$$\mathscr{F}_3(z) = \frac{zF(z)}{\sqrt{1-z^2}(z+\delta_m)(z^2+\delta_m^2)(z^4-\delta_n^4)} \quad \text{for } z = \delta_m, \tag{34c}$$

$$\mathscr{F}_4(z) = \frac{zF(z)}{\sqrt{1-z^2}(z+\mathrm{i}\delta_m)(z^2-\delta_m^2)(z^4-\delta_n^4)} \quad \text{for } z = \mathrm{i}\delta_m. \tag{34d}$$

Further

$$\begin{aligned}\mathrm{Re}⨍_0^1 \frac{xF(x)\,\mathrm{d}x}{\sqrt{1-x^2}(x^4-\delta_n^4)(x^4-\delta_m^4)} &= \int_1^\infty \frac{x\,\mathrm{Im}\,F(x)\,\mathrm{d}x}{\sqrt{x^2-1}(x^4-\delta_n^4)(x^4-\delta_m^4)} \\ &+ Re\{\pi\mathrm{i}[\mathscr{F}_1(\delta_n)+\mathscr{F}_2(\mathrm{i}\delta_n)+\mathscr{F}_3(\delta_m)+\mathscr{F}_4(\mathrm{i}\delta_m)]\}\end{aligned} \tag{35}$$

is obtained from equation (33), provided that $\mathrm{Re}\,F(\mathrm{i}y) = 0$ for $y \in \mathbb{R}$, where

$$\begin{aligned}F(x) &= s^2\delta_n\delta_m G_1(sx_n)G_1(sx_m)\mathrm{J}_0(s\beta x)N_0(s\beta x) \\ &+ \delta_n\delta_m G_1(x_n)G_1(x_m)\mathrm{J}_0(\beta x)\mathrm{N}_0(\beta x) \\ &- s\delta_n\delta_m[G_1(sx_n)G_1(x_m) + G_1(x_n)G_1(sx_m)]\mathrm{J}_0(\beta x)\mathrm{N}_0(s\beta x) \\ &- s^2[\delta_n G_1(sx_n)G_0(sx_m) + \delta_m G_0(sx_n)G_1(sx_m)]x\mathrm{J}_0(s\beta x)\mathrm{N}_1(s\beta x) \\ &+ s[\delta_n G_1(sx_n)G_0(x_m) + \delta_m G_0(x_n)G_1(sx_m)]x\mathrm{J}_1(\beta x)\mathrm{N}_0(s\beta x) \\ &+ s[\delta_n G_1(x_n)G_0(sx_m) + \delta_m G_0(sx_n)G_1(x_m)]x\mathrm{J}_0(\beta x)\mathrm{N}_1(s\beta x) \\ &- s[G_0(x_n)G_0(sx_m) + G_0(sx_n)G_0(x_m)]x^2\mathrm{J}_1(\beta x)\mathrm{N}_1(s\beta x) \\ &- [\delta_n G_1(x_n)G_0(x_m) + \delta_m G_0(x_n)G_1(x_m)]x\mathrm{J}_0(\beta x)\mathrm{N}_1(\beta x) \\ &+ s^2 G_0(sx_n)G_0(sx_m)x^2\mathrm{J}_1(s\beta x)\mathrm{N}_1(s\beta x) \\ &+ G_0(x_n)G_0(x_m)x^2\mathrm{J}_1(\beta x)\mathrm{N}_1(\beta x).\end{aligned} \tag{36}$$

The active sound power consists of two parts, i.e., non-oscillating $\tilde{\mathscr{P}}_{a,nm}$ and oscillating $\tilde{\tilde{\mathscr{P}}}_{a,nm}$. The residues in the first order poles provide a contribution to the former part only. The contribution is

$$\operatorname{Re}\{\pi\mathrm{i}\,[\mathscr{F}_1(\delta_n)+\mathscr{F}_2(\mathrm{i}\delta_n)+\mathscr{F}_3(\delta_m)+\mathscr{F}_4(\mathrm{i}\delta_m)]\}=\frac{\pi}{4}\frac{1}{\delta_n^4-\delta_m^4}$$

$$\times\left\{\frac{1}{\delta_n^2}\left[\frac{-\operatorname{Im}F(\delta_n)}{\sqrt{1-\delta_n^2}}+\frac{\operatorname{Im}F(\mathrm{i}\delta_n)}{\sqrt{1+\delta_n^2}}\right]+\frac{1}{\delta_m^2}\left[\frac{\operatorname{Im}F(\delta_m)}{\sqrt{1-\delta_m^2}}-\frac{\operatorname{Im}F(\mathrm{i}\delta_m)}{\sqrt{1+\delta_m^2}}\right]\right\}$$

$$=\frac{(2\beta)^{-1}}{\delta_n^4-\delta_m^4}\left\{\frac{1}{\delta_n^2}\left(\frac{-1}{\sqrt{1-\delta_n^2}}+\frac{1}{\sqrt{1+\delta_n^2}}\right)[\delta_n G_1(x_n)G_0(x_m)+s\delta_m G_1(sx_m)G_0(sx_n)]\right.$$

$$\left.+\frac{1}{\delta_m^2}\left(\frac{1}{\sqrt{1-\delta_m^2}}-\frac{1}{\sqrt{1+\delta_m^2}}\right)[\delta_m G_1(x_m)G_0(x_n)+s\delta_n G_1(sx_n)G_0(sx_n)]\right\}, \tag{37}$$

where

$$\operatorname{Im}F(\delta_n)=2(\pi\beta)^{-1}\,[\delta_n G_1(x_n)G_0(x_m)+\delta_m s G_1(sx_m)G_0(sx_n)], \tag{38a}$$

$$\operatorname{Im}F(\delta_m)=2(\pi\beta)^{-1}\,[\delta_m G_1(x_m)G_0(x_n)+\delta_n s G_1(sx_n)G_0(sx_m)], \tag{38b}$$

$$F(\mathrm{i}\delta_n)=2\mathrm{i}(\pi\beta)^{-1}\,[\delta_n G_1(x_n)G_0(x_m)+s\delta_m G_1(sx_m)G_0(sx_n)], \tag{38c}$$

$$F(\mathrm{i}\delta_m)=2\mathrm{i}(\pi\beta)^{-1}\,[\delta_m G_1(x_m)G_0(x_n)+s\delta_n G_1(sx_n)G_0(sx_m)]. \tag{38d}$$

Several asymptotic formulae, valid for large arguments only, are used to compute the integral from equation (35) within the infinite limits $(1,+\infty)$

$$\sqrt{s}\mathrm{J}_0(u)\mathrm{N}_1(su)\sim-(\pi u)^{-1}[\sin(s+1)u+\cos(s-1)u], \tag{39a}$$

$$\sqrt{s}\mathrm{J}_0(u)\mathrm{N}_0(su)\sim(\pi u)^{-1}[\sin(s-1)u-\cos(s+1)u], \tag{39b}$$

$$\sqrt{s}\mathrm{J}_1(u)\mathrm{N}_0(su)\sim(\pi u)^{-1}[-\sin(s+1)u+\cos(s-1)u], \tag{39c}$$

$$\sqrt{s}\mathrm{J}_1(u)\mathrm{N}_1(su)\sim(\pi u)^{-1}[\sin(s-1)u+\cos(s+1)u]. \tag{39d}$$

The integral from equation (35) is computed by means of the stationary phase method (cf., reference [22]). The integral provide contribution to the oscillating part of the sound power only, with the exception for the case when the arguments of Bessel and Neumann functions are identical in equation (39a), i.e., $s=1$, which transforms equation (39a) to the form

$$\mathrm{J}_0(u)\mathrm{N}_1(u)\sim-(\pi u)^{-1}[1+\sin 2u],\quad u\in\{\beta x,\,s\beta x\}. \tag{40}$$

Equation (40) consists of two terms: $-1/(\pi u)$ and $-\sin 2u/(\pi u)$, after integration in equation (35) the former represents the non-oscillating contribution and the latter the oscillating contribution to the active mutual sound power. The non-oscillating contribution may be computed by means of the following formula:

$$\int_1^\infty\frac{x\,\mathrm{d}x}{\sqrt{x^2-1\,(x^4-\delta_n^4)(x^4-\delta_m^4)}}=\frac{\pi}{2}\frac{1}{\delta_n^4-\delta_m^4}$$

$$\times\left[\frac{1}{\delta_n^2}\left(\frac{1}{2\sqrt{1-\delta_n^2}}-\frac{1}{2\sqrt{1+\delta_n^2}}\right)-\frac{1}{\delta_m^2}\left(\frac{1}{2\sqrt{1-\delta_m^2}}-\frac{1}{2\sqrt{1+\delta_m^2}}\right)\right]. \tag{41}$$

The sum of all the non-oscillating contributions provides

$$\bar{\mathscr{P}}_{a,nm} = \frac{(\delta_n a_{nm} - \delta_m a_{mn})\,\delta_n^2\delta_m^2}{\beta(\delta_n^4 - \delta_m^4)} \times\left[\frac{1}{\delta_n^2}\left(\frac{1}{\sqrt{1-\delta_n^2}} - \frac{1}{\sqrt{1+\delta_n^2}}\right) + \frac{1}{\delta_m^2}\left(\frac{1}{\sqrt{1-\delta_m^2}} - \frac{1}{\sqrt{1+\delta_m^2}}\right)\right], \tag{42}$$

which is the non-oscillating part of the active mutual sound power, and

$$a_{nm} = (g_n g_m)^{-1}\,[sG_1(sx_n)\,G_0(sx_m) - G_1(x_n)\,G_0(x_m)]. \tag{43}$$

The sum of all the oscillating contributions from the integral provides the oscillating part of the active mutual sound power

$$\tilde{\mathscr{P}}_{a,nm} = \frac{q_n q_m}{\beta\sqrt{\pi\beta}}\{\beta_1\cos w_1 + \beta_2\sin w_1 + \sqrt{s}\Big[\alpha_1\cos w_2 + \alpha_2\sin w_2 + \sqrt{\frac{2}{s-1}}(\gamma_1\cos w_3 - \gamma_4\sin w_3) + \sqrt{\frac{2}{s+1}}(\gamma_3\cos w_4 - \gamma_2\sin w_4)\Big]\Big\}, \tag{44}$$

where

$$q_n = \sqrt{2}\,g_n^{-1}\delta_n^2(1-\delta_n^4)^{-1}, \quad \beta_1 = G_0(x_n)\,G_0(x_m) - \delta_n\delta_m G_1(x_n)\,G_1(x_m),$$

$$\beta_2 = \delta_m G_1(x_m)\,G_0(x_n) + \delta_n G_1(x_n)\,G_0(x_m), \quad w_1 = 2\beta + \pi/4, \quad w_2 = 2s\beta + \pi/4,$$

$$\alpha_1 = G_0(sx_n)\,G_0(sx_m) - \delta_n\delta_m G_1(sx_n)\,G_1(sx_m), \quad w_3 = (s-1)\beta + \pi/4,$$

$$\alpha_2 = \delta_m G_1(sx_m)\,G_0(sx_n) + \delta_n G_1(sx_n)\,G_0(sx_m), \quad w_4 = (s+1)\beta + \pi/4,$$

$$\gamma_1 = \delta_n G_1(sx_n)\,G_0(x_m) - \delta_n G_1(x_n)\,G_0(sx_m) - \delta_m G_1(x_m) G_0(sx_n) + \delta_m G_1(sx_m)\,G_0(x_n),$$

$$\gamma_2 = \delta_n G_1(sx_n)\,G_0(x_m) + \delta_n G_1(x_n)\,G_0(sx_m) + \delta_m G_1(x_m) G_0(sx_n) + \delta_m G_1(sx_m)\,G_0(x_n),$$

$$\gamma_3 = \delta_n\delta_m[G_1(sx_n)\,G_1(x_m) + G_1(sx_m)\,G_1(x_n)] - G_0(sx_n)G_0(x_m) - G_0(sx_m)\,G_0(x_n),$$

$$\gamma_4 = \delta_n\delta_m[G_1(sx_n)\,G_1(x_m) + G_1(sx_m)\,G_1(x_n)] + G_0(sx_n)G_0(x_m) + G_0(sx_m)\,G_0(x_n).$$

The sum of both parts, from equations (42) and (44), provides the asymptotic formula for the standardized active mutual sound power

$$\mathscr{P}_{a,nm} = \bar{\mathscr{P}}_{a,nm} + \tilde{\mathscr{P}}_{a,nm} + \mathscr{O}(\delta_n^2\delta_m^2\beta^{-3/2}), \tag{45}$$

where the term $\mathscr{O}(\cdots)$ represents the approximation error (cf., the modulus and phase cosine of the mutual sound power shown in Figures 3–5).

The determination of the standardized reactive mutual sound power requires computing the integral in equation (28) within the infinite limits $x \in (1, +\infty)$, along the path shown in Figure 1, which leads to the transformation $1/\sqrt{1-x^2} = -\mathrm{i}/\sqrt{x^2-1}$ and results in

$$\mathscr{P}_{r,nm} = \int_1^{+\infty} \Psi_n(x)\,\Psi_m(x)\,\frac{x\,\mathrm{d}x}{\sqrt{x^2-1}}. \tag{46}$$

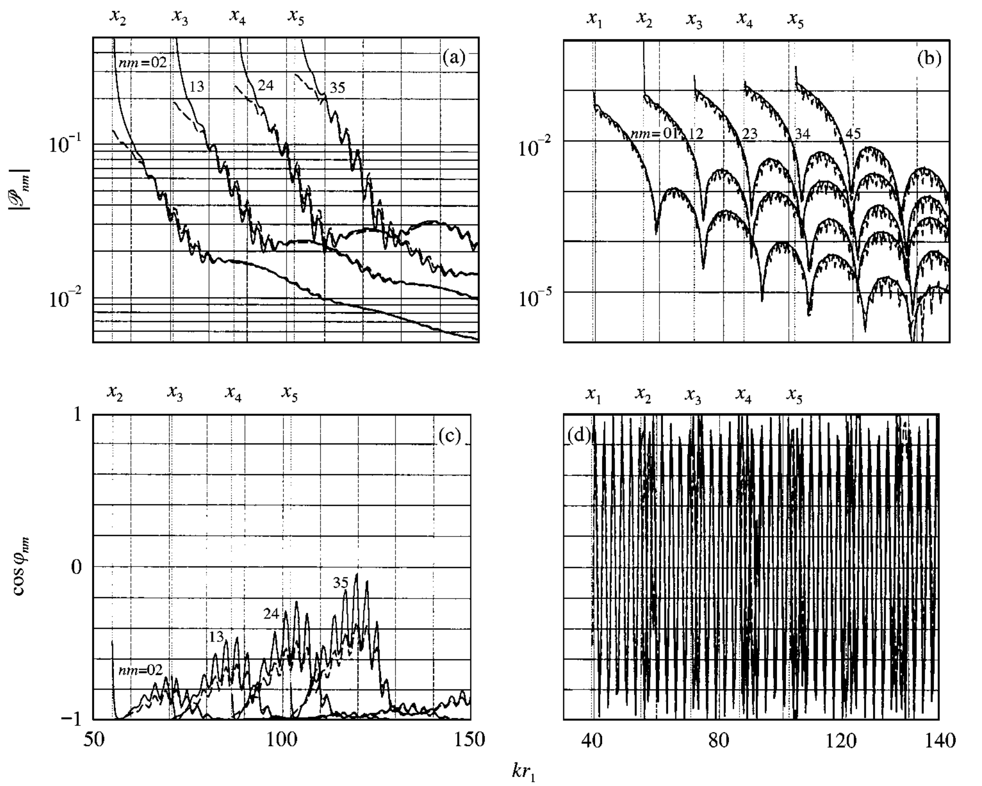

Figure 3. The standardized mutual sound power $\mathscr{P}_{nm}$ for $s = 1{\cdot}2$: (a), (b) the modulus $|\mathscr{P}_{nm}|$ and (c), (d) the phase cosine $\cos\varphi_{nm}$. All the dashed curves in all the plots presented in Figure 3–5 are obtained from the integral formulae, and all the solid curves from the asymptotic formulae.

Further asymptotic approximations for the products of Bessel's functions are used, under,

$$\mathrm{J}_0^2(u) \sim (\pi u)^{-1}[1 + \sin 2u], \quad \mathrm{J}_1^2(u) \sim (\pi u)^{-1}[1 - \sin 2u], \tag{47a}$$

$$\sqrt{s}\mathrm{J}_1(su)J_1(u) \sim (\pi u)^{-1}[\cos(s-1)u - \sin(s+1)\,u],$$

$$\sqrt{s}\mathrm{J}_0(su)J_0(u) \sim (\pi u)^{-1}[\cos(s-1)u + \sin(s+1)\,u],$$

$$\sqrt{s}\mathrm{J}_0(su)J_1(u) \sim -(\pi u)^{-1}[\sin(s-1)u + \cos(s+1)\,u],$$

$$\sqrt{s}\mathrm{J}_1(su)J_0(u) \sim (\pi u)^{-1}[\sin(s-1)u - \cos(s+1)\,u]. \tag{47b}$$

They are valid for large arguments only. The formulae

$$\frac{1}{(x^4-\delta_n^4)(x^4-\delta_m^4)} = \frac{1}{\delta_n^4-\delta_m^4}\left[\frac{1}{2\delta_n^2}\left(\frac{1}{x^2-\delta_n^2} - \frac{1}{x^2+\delta_n^2}\right)\right.$$
$$\left. - \frac{1}{2\delta_m^2}\left(\frac{1}{x^2-\delta_m^2} - \frac{1}{x^2+\delta_m^2}\right)\right], \quad \frac{x^2}{x^2 \pm \delta_n^2} = 1 \pm \frac{\delta_n^2}{x^2 \mp \delta_n^2}, \tag{48a}$$

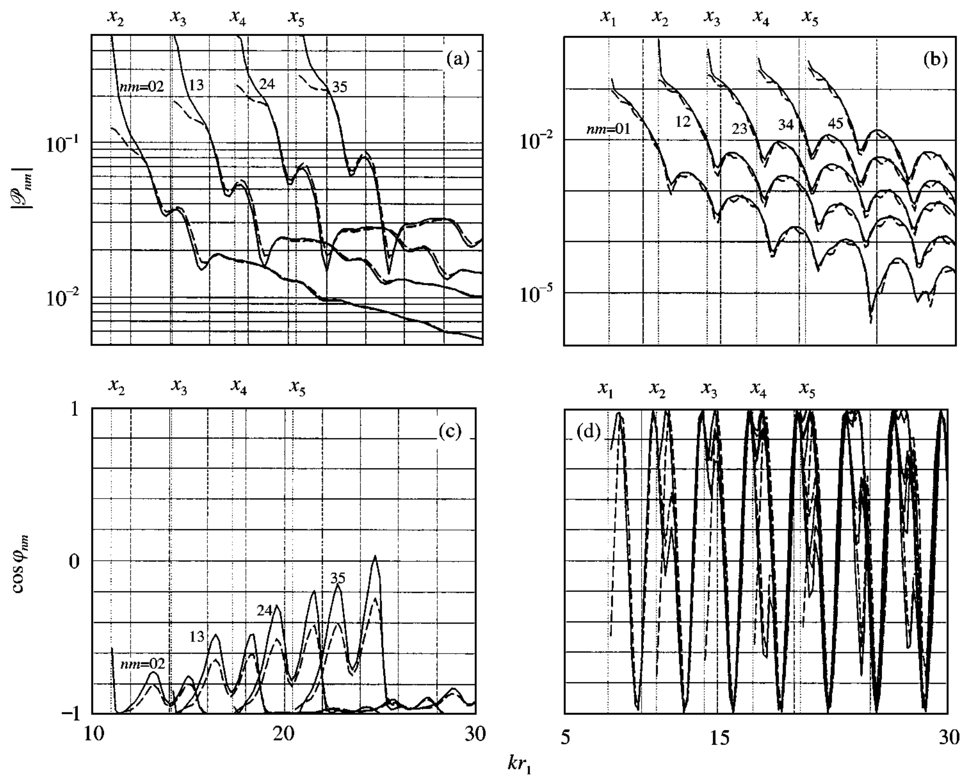

Figure 4. The standardized mutual sound power $\mathscr{P}_{nm}$ for $s = 2$: (a), (b) the modulus $|\mathscr{P}_{nm}|$ and (c), (d) the phase cosine $\cos\varphi_{nm}$.

$$\int_1^{+\infty} \frac{\mathrm{d}x}{\sqrt{x^2-1}(x^2-t^2)} = \frac{\arcsin t}{t\sqrt{1-t^2}}, \quad \int_1^{+\infty} \frac{\mathrm{d}x}{\sqrt{x^2-1}(x^2+t^2)} = \frac{\operatorname{asinh} t}{t\sqrt{1+t^2}}, \quad t<1 \tag{48b}$$

are used for computing the non-oscillating part $\bar{\mathscr{P}}_{r,nm}$ of integral (46). The non-oscillating part $\bar{\mathscr{P}}_{r,nm}$ of the reactive mutual sound power can be expressed as

$$\bar{\mathscr{P}}_{r,nm} = \frac{2(\pi\beta)^{-1}}{\delta_n^4-\delta_m^4}\left\{\delta_m^2\left[\delta_m a'_{nm} + \delta_n a''_{nm})\frac{\arcsin\delta_n}{\sqrt{1-\delta_n^2}} - (\delta_m a'_{nm} - \delta_n a''_{nm})\frac{\operatorname{asinh}\delta_n}{\sqrt{1+\delta_n^2}}\right]\right.$$

$$\left. - \delta_n^2\left[(\delta_n a'_{nm} + \delta_m a''_{nm})\frac{\arcsin\delta_m}{\sqrt{1-\delta_m^2}} - (\delta_n a'_{nm} - \delta_m a''_{nm})\frac{\operatorname{asinh}\delta_m}{\sqrt{1+\delta_m^2}}\right]\right\}, \tag{49}$$

where

$$a'_{nm} = a'_{mn} = (g_n g_m)^{-1}[sG_1(sx_n)G_1(sx_m) + G_1(x_n)G_1(x_m)],$$

$$a''_{nm} = a''_{mn} = (g_n g_m)^{-1}[sG_0(sx_n)G_0(sx_m) + G_0(x_n)G_0(x_m)].$$

Further denotations are introduced

$$d_1 = a'_{nm}\delta_m + a''_{nm}\delta_n, \quad d_2 = a'_{nm}\delta_m - a''_{nm}\delta_n, \quad d_3 = a'_{nm}\delta_n + a''_{nm}\delta_m, \quad d_4 = a'_{nm}\delta_n - a''_{nm}\delta_m$$

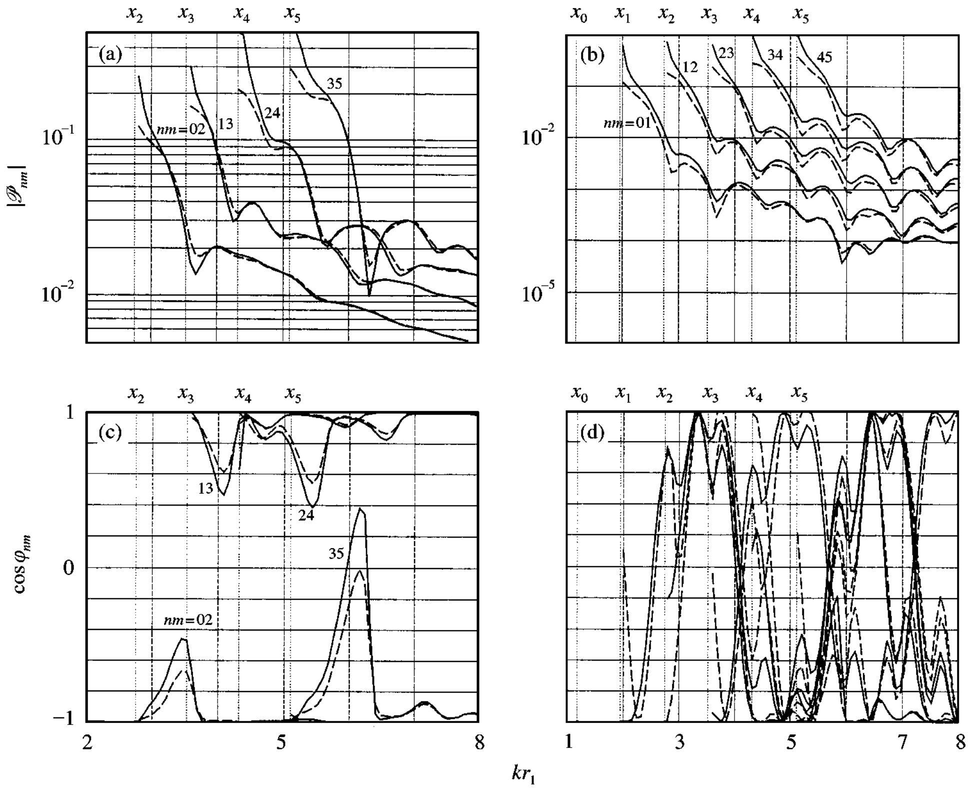

Figure 5. The standardized mutual sound power $\mathscr{P}_{nm}$ for $s = 5$: (a), (b) the modulus $|\mathscr{P}_{nm}|$ and (c), (d) the phase cosine $\cos \varphi_{nm}$.

to present equation (49) in its final form

$$\bar{\mathscr{P}}_{r,nm} = \frac{2(\pi\beta)^{-1}}{\delta_n^4 - \delta_m^4}\left[\delta_m^2\left(d_1\frac{\arcsin\delta_n}{\sqrt{1-\delta_n^2}} - d_2\frac{\operatorname{asinh}\delta_n}{\sqrt{1+\delta_n^2}}\right) - \delta_n^2\left(d_3\frac{\arcsin\delta_m}{\sqrt{1-\delta_m^2}} - d_4\frac{\operatorname{asinh}\delta_m}{\sqrt{1+\delta_m^2}}\right)\right]. \quad (50)$$

The asymptotic formula

$$\int_1^\infty \frac{x^r e^{ibx}\,dx}{\sqrt{x^2-1}(x^4-\delta_n^4)(x^4-\delta_m^4)} = \sqrt{\frac{\pi}{2b}}\left[\frac{1}{(1-\delta_n^4)(1-\delta_m^4)} + \mathscr{O}(b^{-1})\right]e^{i(b+\pi/4)}, \quad (51)$$

is obtained by the stationary phase method (cf., reference [22]), where $r = 0, 1, 2$. The formula is used for computing the oscillating part $\tilde{\mathscr{P}}_{r,nm}$ of integral (46) to present it in its final form as

$$\tilde{\mathscr{P}}_{r,nm} = \frac{q_n q_m}{\beta\sqrt{\pi\beta}}\left\{-\beta_1\sin w_1 + \beta_2\cos w_1 - \sqrt{s}\left[\alpha_1\sin w_2 - \alpha_2\cos w_2 + \sqrt{\frac{2}{s-1}}(\gamma_1\sin w_3 + \gamma_4\cos w_3) + \sqrt{\frac{2}{s+1}}(\gamma_3\sin w_4 + \gamma_2\cos w_4)\right]\right\}. \quad (52)$$

The sum of equations (50) and (52) provides the expression representing the standardized reactive mutual sound power

$$\mathscr{P}_{r,nm} = \bar{\mathscr{P}}_{r,nm} + \tilde{\mathscr{P}}_{r,nm} + \mathcal{O}(\delta_n^2 \delta_m^2 \beta^{-3/2}). \tag{53}$$

The term $\mathcal{O}(\cdots)$ in equation (53) represents the approximation error analogous to that in equation (45) (cf., the modulus and phase cosine of the mutual sound power plotted in Figures 3–5). The asymptotic formulae (45) and (53) have now been presented in their elementary forms. They are valid if the conditions $k > k_n$, $k > k_m$, and $kr_1 > 10$ are satisfied, i.e., for the acoustically fast waves.

3.3. *Some numerical results*

The frequency characteristics presented herein indicate that the closer the indices nm of the interacting mode pair the greater is the mutual sound power generated. A particular case occurs when the distance between the indices is equal to zero, i.e., $n = m$, which results in the sound power for individual *in vacuo* modes providing the greatest contribution to the total sound power of the plate. This case has been discussed earlier in reference [5].

If the distance between indices nm is an odd number (cf., sub-figures (b) and (d) in Figures 3–5), the non-oscillating part vanishes and the oscillating part is well marked. On the other hand, if the distance is an even number (cf., sub-figures (a) and (c) in Figures 3–5), the non-oscillating part is well marked and the oscillating part has a smaller value of its amplitude for different values of the geometric parameter s, than in the former case. This is valid for both parts of the mutual sound power, active and reactive.

Some characteristic oscillations in the oscillating parts of the mutual sound power can be observed in all the frequency characteristics presented herein. They are induced by strong mechanical interactions between both rigidly clamped edges of the plate. The greater the similarity of the plate's shape to an annulus, i.e., $r_1 \to r_2$, the greater is the number of oscillations per frequency unit which decreases when the plate approximates a circle, i.e., $r_1 \to 0$, $r_2 = const$. Both oscillating and non-oscillating parts have been isolated in the asymptotic formulae, which is not possible while employing any other known analytic method. The analysis of both parts confirms some earlier assumptions, e.g., that the non-oscillating part vanishes if the distance between the indices of the interacting mode pair is an odd number.

The asymptotic formulae for the standardized mutual sound power, active and reactive, are valid for the acoustically fast waves presented in their elementary forms. This means that the formulae make possible some fast and satisfactorily precise numerical computations for practical use. This is the main advantage of the formulae as compared with the formulae for the magnitudes under consideration as e.g., those presented in the form of multiple series (cf., reference [2]). Their main disadvantage is that they are valid for the acoustically fast waves only. In the case of the acoustically slow waves, i.e., $k < k_n$ and $k < k_m$, the integral formulae (30) and (46) or any other valid analytical formulae must be used. No analytical formulas for the reactive mutual sound power of a clamped annular plate have been reported before. The formulae presented in this paper together with those given in reference [5] can be used as the complete basis for determination of the total sound power of a forced, damped and clamped annular plate in fluid.

4. CONCLUDING REMARKS

So far, no asymptotic formulae in their elementary form have been presented for the mutual active and reactive sound power for an annular plate. A circular plate is a simpler case than an annular plate is the sense of both physics and computations. The complexity of the annular plate's behavior is caused mainly by its boundary conditions, meaning that both edges of the plate are rigidly clamped to the baffle, resulting in several modifications of the plate's motion and radiation, when compared with a circular plate. Moreover, the asymptotic methods used for an annular plate must be more general than those used for a circular plate.

The asymptotic formulae presented herein can be used for some fast and satisfactorily precise numerical computations of the total sound power. The formulae together with the results presented in reference [5] make a complete basis for this. The formulae consist of some elementary expressions only, and therefore they make possible some definitely faster numerical computations than in the case of using any other analytical methods. Their main disadvantage is that they are valid for high frequencies only, i.e., $k_n > k$ and $k_m > k$. For the other frequencies it is necessary to use the other currently available computationally slower methods such as those given in reference [12] or the integral formulae presented herein together with those given in reference [5].

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†State Committee for Scientific Research, 1/3 Wspólna Str., 00-529 Warsaw 53, Poland.

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